Classroom

In this section of *Resonance,* we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

The Landau Theory of Phase Transitions: A Mechanical Analog

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The Landau theory of phase transitions occupies a centerpiece position in physics. We illustrate the theory by a pedagogical example at the pre-university level. In the example a bead of mass m is threaded on a ring which is set rotating about a vertical diameter. The dynamics of the bead mimics key features of the Landau theory.

1. Introduction

2008 was the centenary year of the great Russian physicist L D Landau. Born on Jan 22, 1908, Landau grew up to be an eminent scientist who made seminal contributions to all branches of theoretical physics. Landau was also an outstanding scholar and teacher of physics. His accomplishments include the co-discovery of the density matrix method in quantum mechanics, quantum mechanical theory of diamagnetism, the theory of second order phase transitions, the mean-field theory of superconductivity, the explanation of Landau damping in plasma physics, the Landau pole in quantum electro-

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dynamics, and the two-component theory of neutrinos. Many of us are familiar with 'Landau–Lifshitz' series of volumes in Theoretical Physics [1, 2]. Several physics terms bear his name.

Landau received many honours. In the Soviet Union he was directly elected as a member of the Academy of Science and was given the title of Hero of Socialist Effort. He was awarded the 1962 Nobel Prize for Physics. Along with Vitalyn Ginzburg, Landau made a milestone contribution to the theory of second order phase transition [3]. The application of this theory is far reaching and ranges from chemical sciences to particle physics. In the present article we shall try to understand the essence of this theory by studying a simple mechanical model.

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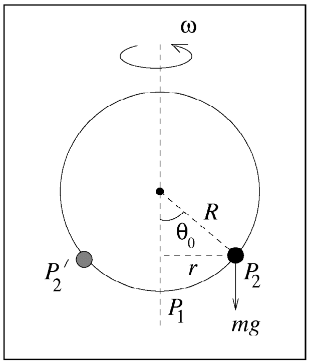
2. The Model

Our model consists of a ring of radius R with a small bead of mass m threaded on it [4]. The ring is set rotating about its vertical diameter with angular velocity ω as shown in *Figure* 1. There is no friction. One begins with small ω which gradually increases. It is observed that the bead continues to be at the lowest position P_1 until a certain critical angular velocity ω_c is attained. The bead slides up as ω increases beyond ω_c. The figure depicts a typical position of the bead at P_2 (for $\omega > \omega_c$). The behavior can be understood on the basis of potential-energy diagram of the bead. For $\omega < \omega_c$, the potential-energy diagram has a single minimum while for $\omega > \omega_c$ it develops a double minimum (see *Figure* 5 on p.709). This is analogous to the free energy diagram of a magnetic system undergoing second order phase transition. For a temperature $T < T_c$, the free energy with respect to the magnetization M has a single minimum, whereas for $T > T_c$, it has two minima. Here T_c is the critical temperature and is called the 'Curie' temperature. We shall explore and exploit this analogy gainfully in this article.

Figure 1. The bead on a rotating ring.



3. Analogies

3.1 *The 'Phase Transition'*

Let us analyze the dynamics of the bead on the rotating ring in the non-inertial frame. We shall employ polar co-ordinates $\{r, \theta\}$. We will find that it is useful to express our answers in terms of $\omega_{\mathrm{c}} = \sqrt{g/R}$, where g is the magnitude of acceleration due to gravity. Note that in the non-inertial frame the bead will appear stationary at P_2. We shall designate this steady state position as 'equilibrium' although strictly speaking this is a misnomer.

The free body diagram of the bead in the non-inertial frame is shown in *Figure* 2. It is clear from the figure that the tangential and radial components of the force (F_θ and F_{r}) are respectively:

$$F_\theta = F_{\mathrm{cf}} \cos\theta - mg \sin\theta; \qquad (1)$$

$$F_{\mathrm{r}} = N - mg\cos\theta - m\omega^2 R \sin^2\theta. \qquad (2)$$

Here $F_{\mathrm{cf}}(= m\omega^2 R \sin\theta_0)$ is the centrifugal force and N is the normal force on the bead. For 'equilibrium', the tangential component (F_θ) vanishes. Hence

$$\cos\theta_0 = \frac{g}{\omega^2 R} = \frac{\omega_{\mathrm{c}}^2}{\omega^2}\ , \qquad (3)$$

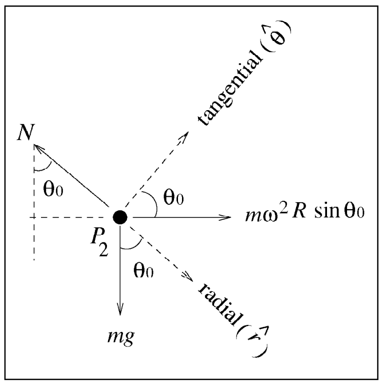


Figure 2. Free body diagram of the bead in 'equilibrium' for $\omega > \omega_c$.

where θ_0 is the 'equilibrium' angle:

$$\theta_0 = \pm \left| \cos^{-1} \frac{\omega_c^2}{\omega^2} \right| \ .$$

The $\pm$ indicates that there are two equivalent positions, P_2 and P_2' (see *Figure* 1). We shall comment on this later after describing *Figure* 5. Note that for $\omega < \omega_c$, equation (3) implies $\cos\theta_0 > 1$. This is clearly unphysical. A little reflection will convince us that $\theta = 0$ for $\omega < \omega_c$. It can be also seen from (1) and (2) that, for $\theta = 0$, $F_\theta = 0$, $N = mg$, i.e., the ring is rotating but the bead is at the bottom (at P_1). This indicates a transition in *Figure* 1 at $\omega = \omega_c$. The analogy with phase transition is strengthened when we examine the behavior of the normal (N) and centrifugal (F_{cf}) forces with ω.

For $\omega < \omega_c$

$$\theta_0 = 0,\ N = mg \text{ and } F_{cf} = 0,$$

whereas for $\omega > \omega_c$

$$N = m\omega^2 R \text{ and } F_{cf} = m\omega^2 R \left[1 - \frac{\omega_c^4}{\omega^4}\right]^{1/2} \ . \qquad (4)$$

The last equation is obtained using (3) and the trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$. The behavior of forces (N and F_{cf}) are plotted in *Figure* 3 where again we see the transition at $\omega = \omega_c$.

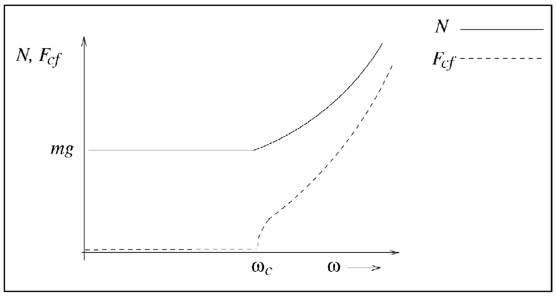


Figure 3. Dependence of normal (N) and centrifugal (F_{cf}) forces on ω. For large values of ω, F_{cf} approaches N.

The role of M is played by θ and temperature is inversely related to ω.

3.2 *The 'Magnetization'*

We next obtain the dependence of the 'equilibrium' angle θ_0 on ω as ω approaches ω_c. For $\omega \to \omega_c^+$, θ_0 is close to zero. Hence on expanding the cosine term in (3), we get

$$\begin{aligned} 1 - \frac{\theta_0^2}{2} &= \frac{\omega_c^2}{\omega^2}, \\ \theta_0 &= \pm\sqrt{2}\left[1 - \frac{\omega_c^2}{\omega^2}\right]^{1/2} . \end{aligned} \qquad (5)$$

Also note from (3) that as $\omega \to \infty$, $\theta_0 \to \pm\pi/2$. This behavior is sketched in *Figure* 4. This plot also has an analogue in phase transition. The magnetization M goes to zero as T goes to T_c in a similar fashion. This is shown in the inset of *Figure* 4, where M_0 is the maximum magnetization. Thus the role of M is played by θ and temperature is inversely related to ω. Increasing temperature is equivalent to decreasing ω. (Note: The critical exponent is 1/2 in our case and also in Landau theory. However, experimentally and in more elaborate theories, the exponent of vanishing magnetization is 1/3).

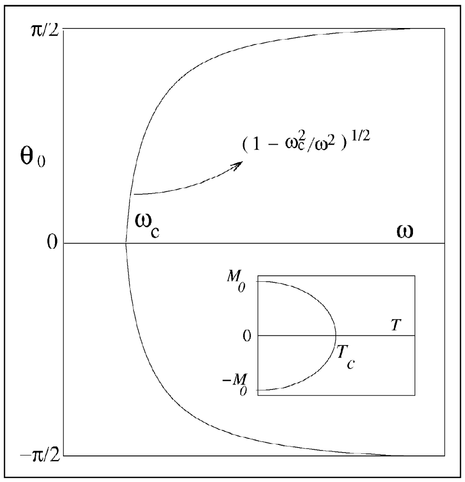


Figure 4. The behavior of θ as function of ω. The inset shows an analogous behavior of magnetization M as a function of temperature T.

3.3 *The 'Free Energy Diagram' and 'Symmetry Breaking'*

It is more instructive to look at the potential energy. Recall that

$$F_\theta = -\frac{1}{R}\frac{\mathrm{d}V(\theta)}{\mathrm{d}\theta} \ .$$

Taking $V(\theta = 0) = 0$, we obtain

$$V(\theta) = mgR\left[(1-\cos\theta) - \frac{\omega^2}{2\omega_\mathrm{c}^2}\sin^2\theta\right] \ . \qquad (6)$$

On expanding the $\cos\theta$ and $\sin\theta$ terms in (6), we get

$$\begin{aligned} V(\theta) &\simeq a(\omega)\theta^2 + b(\omega)\theta^4 \ , \\ a(\omega) &= \frac{mgR}{2}\left(1-\frac{\omega^2}{\omega_\mathrm{c}^2}\right) \ , \\ b(\omega) &\simeq \frac{mgR}{6}\frac{\omega^2}{\omega_\mathrm{c}^2} \ , \end{aligned}$$

for small θ and upto order (θ^4). We get two types of behavior as shown in *Figure* 5. For $\theta = 0, V(\theta) = 0$; it is stable for $\omega < \omega_\mathrm{c}$ and unstable for $\omega > \omega_\mathrm{c}$. For $\omega > \omega_\mathrm{c}$,

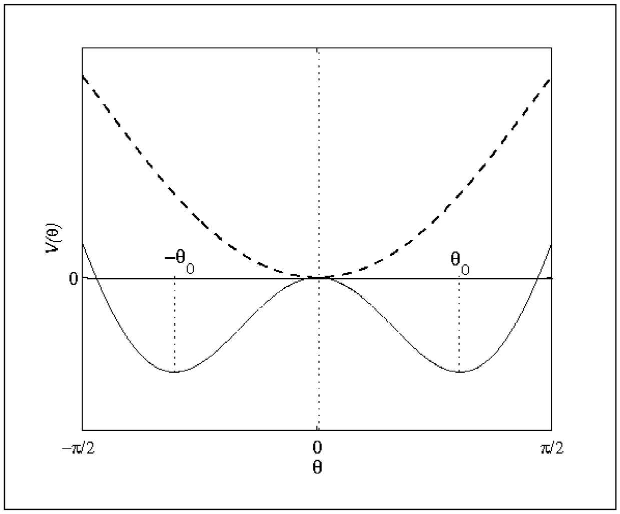


Figure 5. Potential energy for $\omega < \omega_c$ ($\omega > \omega_c$) is depicted by the dotted (solid) line.

The existence of symmetry breaking has played a cardinal role in modern particle physics and condensed matter theory.

θ has two minima at $\pm\,\theta_0$. We can observe this by looking at the second order derivative:

$$V''(\theta) = mgR\cos\theta\left[1-\frac{\omega^2}{\omega_{\rm c}^2}\cos\theta\right] + mgR\frac{\omega^2}{\omega_{\rm c}^2}\sin^2\theta \ . \quad (7)$$

For $\theta = \theta_0 = \pm\cos^{-1}\left(\omega_{\rm c}^2/\omega^2\right)$,

$$V''(\theta_0) = mgR\frac{\omega^2}{\omega_{\rm c}^2}\left(1-\frac{\omega_{\rm c}^4}{\omega^4}\right) > 0 \qquad \text{if } \omega > \omega_{\rm c} \ . \quad (8)$$

Thus $\pm\,\theta_0$ are stable minima. A very important concept in modern physics is symmetry breaking. This is captured by the double minima diagram in *Figure* 5. For $\omega > \omega_{\rm c}$, $\theta = +\theta_0$ and $\theta = -\theta_0$ are equally likely. However the smallest perturbation, say a whiff of air or a vertical axis which is slightly offcentre, will make the bead go towards one of the minima and symmetry will be broken. The existence of symmetry breaking has played a cardinal role in modern particle physics and condensed matter theory.

3.4 *'Critical Slowing Down'*

It is also instructive to determine the frequency of oscillation Ω_0 of the bead about the 'equilibrium' position θ_0.

$$\Omega_0 = \frac{1}{R}\sqrt{\frac{V''(\theta)}{m}} \ . \quad (9)$$

For $\omega < \omega_{\rm c}$, $\theta_0 = 0$, and we obtain from (7) that

$$\Omega_0 = (\omega_{\rm c}^2 - \omega^2)^{1/2} \ . \quad (10)$$

Similarly for $\omega > \omega_{\rm c}$, using (8) we obtain

$$\Omega_0 \;=\; \omega\left(1-\frac{\omega_{\rm c}^4}{\omega^4}\right)^{1/2} \ . \quad (11)$$

The behavior of Ω_0 is sketched in *Figure* 6. The fact that as $\omega \to \omega_{\rm c}^{\pm}$, $\Omega_0 \to 0$ is called 'critical slowing down'.

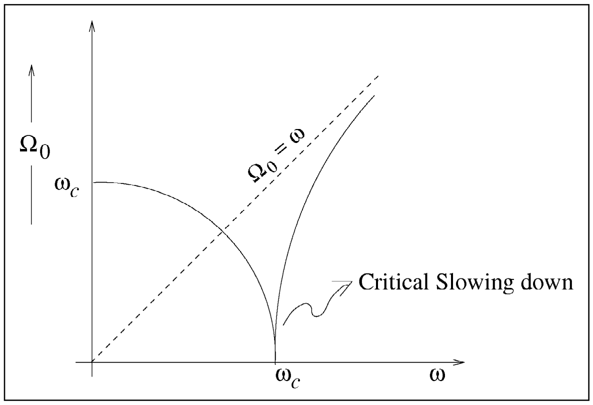


Figure 6. Dependence of the bead's oscillation frequency Ω_0 on the angular speed ω. For large values of ω, Ω_0 approaches the asymptote depicted by the dotted line $\Omega_0 = \omega$.

Similarly near the phase transition, the magnetic moment also becomes sluggish.

3.5 *The 'Susceptibility Divergence'*

An important aspect of phase transition in magnetic systems is the divergence of susceptibility. Interestingly even this aspect is captured by our simple mechanical model. We apply a weak tangential force $f \cos \Omega\, t$ to the bead in 'equilibrium'. The equation of motion is, (see equation (1)),

$$mR\ddot{\theta} = m\omega^2 R \sin\theta \cos\theta - mg \sin\theta + f \cos \Omega t \ .$$

For $\omega < \omega_{\rm c},\ \theta \simeq 0,\ \sin\theta \simeq \theta,\ \cos\theta \simeq 1$ and hence

$$\ddot{\theta} = -\Omega_0^2 \theta + \frac{f}{mR}\ \cos \Omega t \ .$$

Recall that $\Omega_0^2 = \omega_{\rm c}^2 - \omega^2$ from (10). Let us assume the steady state solution $\theta = \theta_0\ \cos \Omega t$. We then obtain

$$\frac{\theta(t)}{f} = \frac{1}{mR}\ \frac{1}{(\Omega_0^2 - \Omega^2)}\ \cos \Omega t \ . \qquad (12)$$

In magnetic systems the role of $f\ \cos \Omega\, t$ is replaced by the time varying external magnetic field and that of θ, as stated earlier, by the magnetization M. Thus θ/f is the 'susceptibility' χ of the bead–ring system. The static susceptibility $\chi(0)$ for $\Omega \to 0$, is θ_0/f.

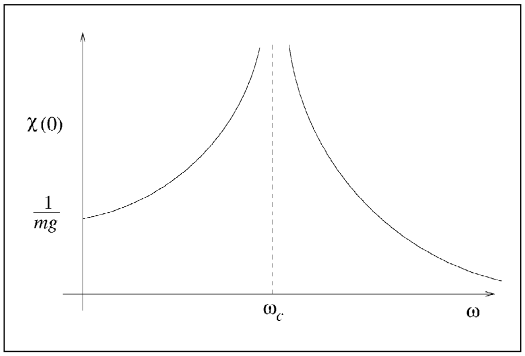


Figure 7. 'Static susceptibility' $\chi(0)$ (= θ_0/f) with ω.

$$\chi(0) = \frac{1}{mR}\frac{1}{\Omega_0^2}$$
$$= \frac{1}{mR}\frac{1}{(\omega_c^2 - \omega^2)} \qquad \omega < \omega_c$$
$$= \frac{1}{mR}\frac{1}{\omega^2\left(1 - \frac{\omega_c^4}{\omega^4}\right)} \qquad \omega > \omega_c \ .$$

and diverges as $\omega \to \omega_c$ as shown in *Figure* 7. The susceptibility in a magnetic system shows the same behavior. Another name for M is the order parameter. In our case the order parameter is θ. Just as the magnetic field 'orders' the magnetization M in magnetic systems, the rotational angular speed ω 'orders' θ.

4. Conclusion

Table I encapsulates the analogy between our mechanical system and a magnetic system undergoing phase transition from ferromagnetic to paramagnetic phase. This analogy should not be overstretched. Our model has a single degree of freedom, viz. angular deflection. The magnetic system is complex and has about 10^{23} degrees of freedom. Our mechanical analysis is in the non-inertial frame; thus what we designate as 'equilibrium' can be misleading. The mechanical model can be quite fruitful in understanding a wide variety of natural phenomenon. We encourage the reader to come up with other mechanical analogues of the Landau theory of phase transitions.

Suggested Reading

[1] L D Landau and E M Lifshitz, *Course of Theoretical Physics*, 10 volumes. Butterworth-Heinemann, 1977.

[2] V L Ginzburg and L D Landau, *On the Theory of Superconductivity*, *Zh. Eksp. Teor. Fiz.*, Vol.20, 1064, 1950; This paper was published in English in the volume: *Landau L. D. Collected Papers*, Oxford: Pergamon Press, pp. 546, 1965.

[3] E M Lifshitz, L. D. Landau's Plain Talk to Students of Physics, *American Journal of Physics*, Vol.45, No.5, pp.415–422, 1977.

[4] Daniel M Greenberger, Esoteric Elementary Particle Phenomena in Undergraduate Physics – Spontaneous Symmetry Breaking and Scale Invariance, *American Journal of Physics*, Vol.46, p.394, 1978.

[5] Indian Institute of Technology - Joint Entrance Examination physics paper in 1993; problem 4(b). This problem covers Section 3.1 without mentioning the delightful analogy with the Landau theory of phase transition.

No.	Mechanical System	Magnetic System	Reference in text
1	Deflection θ	Magnetization M	Section 3.2
2	Increasing angular speed ω	Decreasing temperature T	Section 3.2
3	Critical angular speed ω_c	Critical (Curie) temperature T_c	Section 3.1
4	Exponent of θ is 0.5	Exponent of M is 0.5	Section 3.2
5	$\theta \to \pm\pi/2$	$M \to \pm M_0$	Figue 4 , Section 3.2
6	Symmetry Breaking $\pm\theta_0$ for $\omega > \omega_c$	Symmetry Breaking $\pm M$ for $T < T_c$	Figure 5, Section 3.3
7	Oscillation frequency $\Omega_0 \to 0$ as $\omega \to \omega_c^{\pm}$	Critical slowing down of magnetic moments near T_c	Figure 6, Section 3.4
8	θ/f diverges as $\omega \to \omega_c$	Susceptibility diverges as $T \to T_c$	Figure 7, Section 3.5

Table 1. Analogy between mechanical and magnetic systems.

We note in passing that the derivation in Section 3.1 has been a popular problem for higher secondary school students and has even appeared in the Indian Institute of Technology – Joint Entrance Examination physics paper in 1993. However this wonderful analogy is not even mentioned, thus reducing it to another 'dry' problem. To honour Landau we posed this problem as a challenging award winning problem in the selection camp leading to the selection of the Indian team for the International Physics Olympiad. The winner, Ish Dhand of Chandigarh was given the Indian Physics Association cash award of Rs. 3000/- for the best solution to this problem.

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