Integrating the Measure and Quotient Interpretation of Fractions

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Fractions in the School Curriculum

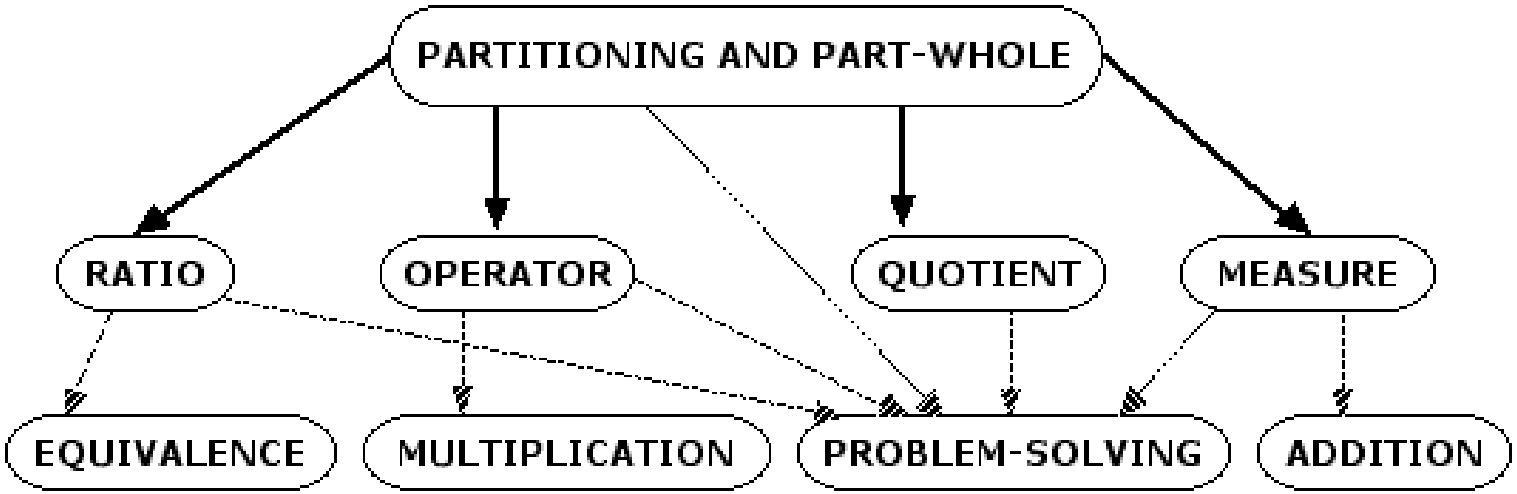
- Fractions are difficult.
- Many children fail to develop a conceptual understanding of fractions.
- One reason is that fraction is a complex concept which is a combination of several subconstructs.

Other sources of difficulty in understanding fractions

- Unfamiliar way of parsing the notation
- Lack of familarity with using the multiplication sign together with the unit fraction for the division operation
- Lack of cultural support for the new units (unit fractions)
- Relative unfamiliarity with the division operation

The Subconstruct Theory

- Proposed by Thomas Kieren in the 1970s
- Many researchers hold that there are five distinct subconstructs: part-whole, measure, quotient, ratio and operator.
- It has been suggested that children should develop an integrated understanding of different subconstructs (Post et al., 1993).
- Other terms used in place of ‘subconstruct’: interpretation, perspective
- The present study is an attempt to develop an integrated understanding of the measure and quotient interpretations.
- Other recent work that attempts to develop a multiple ‘perspective’ approach through teaching intervention: Moseley, 2005.



Ref: Behr et al., 1980, 1981

The part-whole subconstruct

- The traditional school curriculum devotes much time to developing the part-whole subconstruct.
- Considered a good starting point since it is grounded in children's partitioning schemes.
- Interpreted through area, linear or discrete representations.
- Requires children to attend to part-whole and not part-part relation. Hence Piagetian operation of class inclusion necessary.

Difficulties with the part-whole interpretation

- Children often ignore the requirement that parts be equal.
- Children are familiar with counting contexts. Measurement ideas are still developing.
- In counting contexts, the size of a unit may be ignored. For example, when counting the number of people standing in a queue to buy tickets.
- Traditional teaching emphasizes the counting aspect rather than the measurement aspect of the part-whole subconstruct.
- Partitioning scheme is reinforced, but the formation of unit structures in partitioning is underplayed (Lamon 1996).

Emphasizing the measure interpretation

- In our approach, we strengthened the measure interpretation by emphasizing the concept of a unit fraction.
- Unit fractions are frequently overlooked when language support is inadequate.
- For many students who learn in English as a second language, the words ‘fourths’, ‘fifths’, etc. are difficult to speak and to grasp.
- Hence for the fraction $\frac{3}{4}$, the phrases ‘three by four’ or ‘three out of four’ are frequently used.

Advantages of emphasizing unit fractions

- When the unit fractions are explicitly named, it focuses attention on the construction of a new subunit and its relation to the base unit.
- The fraction notation can be unpacked to reveal the composition in terms of unit fractions (elaboration in the symbolic register):

$$\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$

- The unit fractions form a complete sequence that is ordered in terms of size. The obvious analogy with whole numbers helps students grasp the magnitude of unit fractions quickly.
- Using the concept of unit fractions, students are able to reason about fraction magnitudes.

The study

- Part of a larger ongoing study on developing fraction knowledge for dealing with ratio and proportion
- Uses design experiment methodology, with iterations where the same topic is taught to different groups of students.
- The present study focuses on the initial fraction concepts: interpreting the symbol, magnitude and equivalence.
- Two student groups who had prior instruction on fractions in school based on the part-whole interpretation participated in the study.
- In our approach we developed the measure and the quotient interpretation together using students' knowledge of partitioning.

Data

- Here we report students' understanding of fraction magnitude through representation and comparison tasks.
- Data drawn from two groups of students
  - English language: 41 students, avg. age = 10.5 y, 16 sessions of 1.5 hours.
  - Marathi language: 30 students, avg. age = 11 y, 14 sessions of 1.5 hours.
- Data collected: Video records of lessons, pre-, mid- and post-tests
- Ten students (English - 6; Marathi - 4) who were considered weak in their understanding were interviewed.
- The purpose of the interviews was to probe the nature of students' difficulties.

The Approach

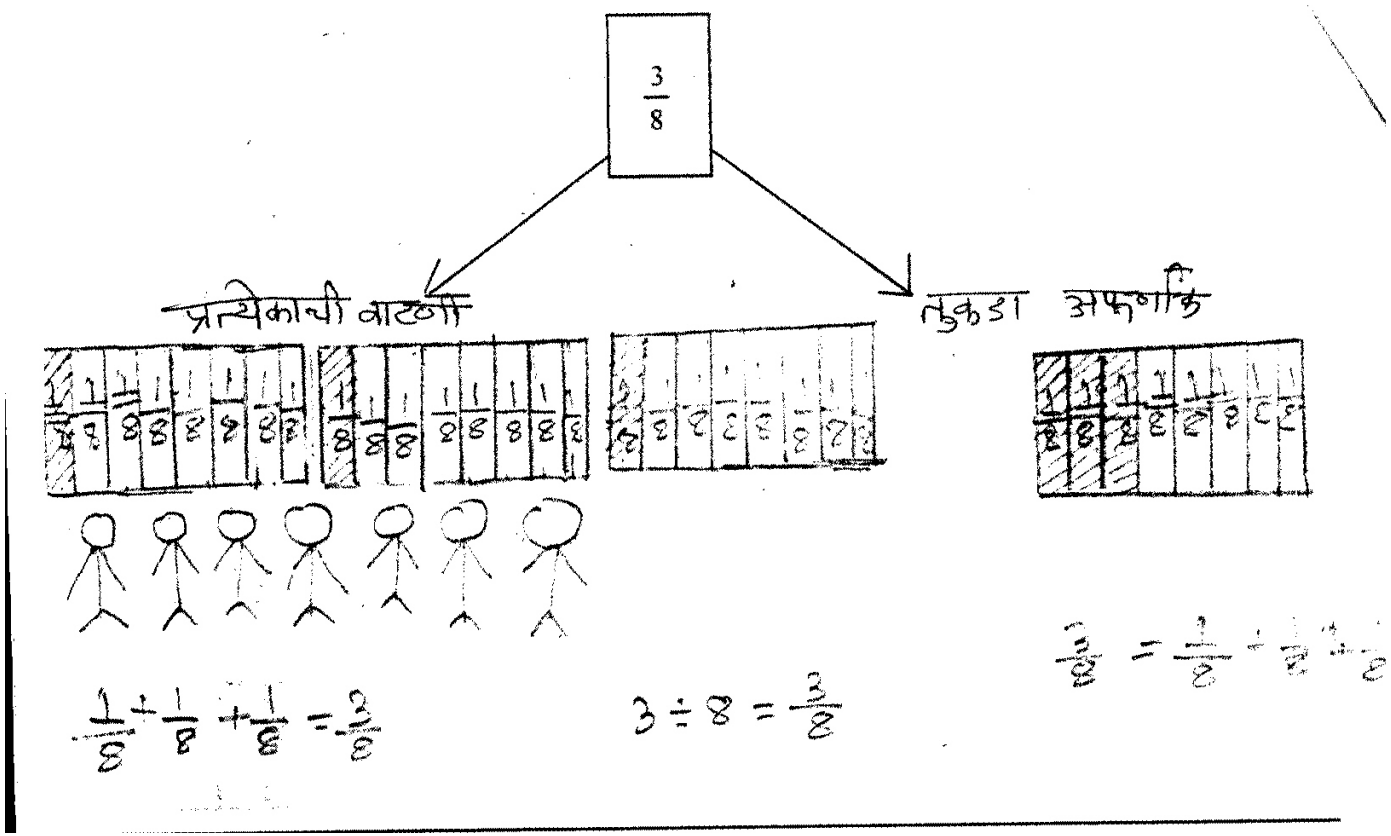
- The teaching unit consisted of two segments:
  - Interpreting, representing and comparing fractions
  - Equivalent fractions and fraction as operator
- We report here students' work from the first segment, which focused largely on representation and comparison tasks.
- This segment formed 9 days of instruction for the English language group and 6 days of instruction for the Marathi group.
- Students learnt to write fractions for measure and share situations. Comparisons were done by interpreting fractions in terms of these situations.

Interpreting fractions

- The need for fractions was established in the measurement context.
- Unit fractions were defined as follows: when a whole is partitioned into equal parts, then each part is a unit fraction represented by $\frac{1}{n}$.
- Unit fractions were also defined as the share obtained when one whole is shared equally among several people.
- Composite fractions were defined as fractions built up from (or measured by) unit fractions. The need for composite fractions arises when we quantify a part that is not a unit fraction.
- The quotient interpretation was introduced through the equal sharing situation adapted from Streefland (1994).

Integrating the two interpretations

The measure and the share interpretations were reconciled by explicitly drawing and comparing diagrams of the two situations.



Overview of students' performance

Writing a fraction for a shaded part (marked parts are unequal, need to be remarked)

Pre-test	Mid-test	Post-test
8.7	31.5	70.1

Writing a fraction for a shaded part (more than a whole)

Pre-test	Mid-test	Post-test
9.9	56.3	59.4

Pictorial representation of improper fraction

Pre-test	Mid-test	Post-test
18	59.8	62

Performance improved marginally when students were asked to write the decomposition of the given fraction in terms of unit fractions.

Pictorial representation of mixed number

Pre-test	Mid-test	Post-test
24.3	73	66.3

Comparison of fractions (all items)

Pre-test	Mid-test	Post-test
37.3	81.3	83.9

Comparison of unit fractions

Pre-test	Mid-test	Post-test
21.2	84.5	83.2

Students' Reasoning about Fractions

- In comparison tasks students drew on both the measure and the share interpretations.
- **Fractions with the same numerator:** *As the number of cakes to share are same, the group where more number of children are there will have a smaller share.*
- *As the number of pieces are same what matters is the size of the unit.*
- **Comparing fractions with half:** Students reasoned that the fraction is equal to half when *the number of cakes is exactly half the number of children* or when *the number of pieces taken is exactly half the total number of pieces.*

Open-ended tasks

- In an open ended task, students were shown a number of fractions and asked to state whatever they knew or were able to find out about the fractions.
- Here students repeatedly used the decomposition of fractions into unit fractions. For example, by decomposing $\frac{5}{4}$ as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, they were able to equate it to to $1\frac{1}{4}$.
- In comparing the fractions 4/5 and 6/7, students reasoned as follows:
  Even though both the fractions 4/5 and 6/7 need one more piece to complete a whole. 4/5 needs one piece of 1/5 and 6/7 needs one piece of 1/7. But 1/5 is more than 1/7 as one cake is shared among 5 children only. Hence 4/5 is away from the whole.

Interview Responses

- Interviews were conducted with ten students selected from the two groups.
- Three tasks were given in the interviews:
  - Pictorial representation of the improper fraction $\frac{14}{9}$
  - Comparison of the fractions $\frac{5}{1}$, $\frac{1}{5}$ and $\frac{5}{5}$
  - Task on equivalent fractions
- Here we discuss responses to the first two tasks.

Representing an improper fraction

- Students were asked to represent the fraction $\frac{14}{9}$ by drawing a picture. After they made a drawing, an alternative drawing (either $\frac{9}{14}$ or $1\frac{5}{9}$) was shown to them and they were asked if it was correct.
- Five of the ten students were confident about the correct response that they made initially.
- All of them used a measure picture, except one student who began with a sharing picture and changed to a measure picture.

- Four students completed the task with difficulty. Two of them started with the sharing picture but changed to a measure picture.
- Two of these four students completed the task by rewriting the improper fraction as a mixed number, but were unsure about the representation of $\frac{14}{9}$.
- One student could not complete the task.

Comparison task

- For the second task of representing and comparing the three fractions: 1/5, 5/5 and 5/1, all except one student completed the comparison task successfully.
- Eight of the nine students could either represent the fraction by a picture or gave a verbal interpretation.
- Seven students used the share interpretation to justify their response.
- Two students, who used the measure interpretation, expressed themselves clearly and confidently.
- One student who reasoned on the basis of both share and measure interpretations was sure that 5/1 is more than 5/5, but was hesitant about drawing a picture or describing precisely how much each quantity was.

Conclusion

- Students exposed to fraction instruction based on the part-whole interpretation can integrate the measure and the quotient perspectives meaningfully with their existing knowledge.
- These new interpretations provide additional resources, which allow them to make sense of improper fractions and mixed numbers.
- These interpretations, especially the quotient or equal sharing interpretation, are a powerful resource in comparing fractions.
- The interviews revealed that students readily draw on both the interpretations, especially on the equal sharing interpretation for comparison.

- In some cases, the equal share meaning did not lead to a clear picture of how much the fraction exactly was. For example, in the case of the fraction $\frac{14}{9}$, it was difficult to draw a picture showing the sharing completely and students either hesitated to do so or withdrew after trying.
- The results taken as a whole indicate that instruction emphasizing the measure and share meaning can positively contribute to students' understanding of fractions and can supplement part-whole understanding, which by itself is inadequate.
- More examples of students' reasoning from additional trials were reported at ICME-11, TSG-3, New Developments and Trends in Mathematics Education at the Lower Secondary Level.