Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

Issues for Research in the Teaching and Learning of Fractions

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Outline

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

- Overview of issues
- Initial fraction knowledge
- Fraction multiplication and division
- Proportional Reasoning

Overview of issues

Some questions about fractions in the curriculum

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Research
Overview Mar
08

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Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

- Why teach fractions?
- When should fractions be taught?
- What should students learn about fractions?
- What *can* students learn as revealed by cognitive/ empirical studies?
- How should instruction be organized for learning fractions better?

Why not remove fractions?

- Fractions are too difficult to learn. They are counter-intuitive and unnatural.
- After adoption of the metric system, they are unnecessary. Decimals are enough and can be learnt independently.

Why teach fractions?

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

- Fractions are a compressed notation for expressing the result of a division operation on any two integers.
- They make multiplication and division operations complete and the procedures easy.
- This makes it easy to handle linear functions, which include proportionality, and which are ubiquitous in the application of mathematics.
- Familiarity with fraction operations necessary for algebraic manipulation.
- Conceptually, they are the first step beyond whole numbers. In the world of measurement it is essential to go beyond whole numbers.
- Decimal numbers, when used as approximations by rounding off, represent intervals, not points. This is different conceptually and harder to understand?

- General agreement that fractions, ratio and proportion belong to the same conceptual network. Vergnaud 88: 'multiplicative conceptual field'
- Studies which investigate the connection between fraction knowledge and understanding of ratio and proportion, especially teaching studies, are few.
- Greer 92: Proportion problems are much harder when one of the given quantities is a fraction:

$$\frac{3}{5} \rightarrow 18; \qquad 2 \rightarrow ?$$

Last week the US National Mathematics Advisory Panel
(end of 'Math Wars'?) recommended:

- Grade 4: Identify and represent fractions and decimals, and compare them.

- Grade 5: Proficient with comparing fractions and decimals and common percents, and with the addition and subtraction of fractions and decimals.

- Grade 6: Proficient with multiplication and division of fractions and decimals.

- Grade 7: Proficient with all operations involving positive and negative fractions. Should be able to solve problems involving percent, ratio and rate and extend this work to proportionality.

- NCERT (NCF 2005) does most of these one year later.

Research finding generally consistent with curriculum sequence.

Many teaching studies do initial work with fractions in grades 3 and 4.

- Grade 3: Ball 96, Kieren 92, Saenz-Ludlow 93, Streefland+ 96, Steffe++* 02, Mack++ 93,
- Grade 4: Davis+* et al 93, Moss 02, Steencken and Maher 03, Bulgar 03, Saenz-Ludlow 03*, Cramer+ et al. 03
- Grade 5: Mack+* 03

'+' – extended to the next year

'*' – used computer software

So when to introduce fractions is not a major issue at present.

Initial Fraction Knowledge

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

"Much is known about how to develop initial fraction knowledge."

Key pieces of knowledge

- Situations for fraction learning
- Fraction Subconstructs
- Schemes underlying fraction understanding
- Semantic analysis (Behr et al. 92)
- Students' informal reasoning (Mack, 1990, 2001)

How to introduce the fraction concept and notation

Introducing fraction situations and symbol

- Fractions answer questions like 'How much? What part?'.
- Begin with familiar fractions: $\frac{1}{2}$, $\frac{1}{4}$, etc.
- Strengthen Equi-partitioning scheme.
- Use sharing situations early.
- Apply Lesh translation model.
- Introduce variability in the manipulatives (Dienes).
- Strengthen unitizing scheme.
- Build up knowledge of the various fraction subconstructs.

Lesh's translation model

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Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning
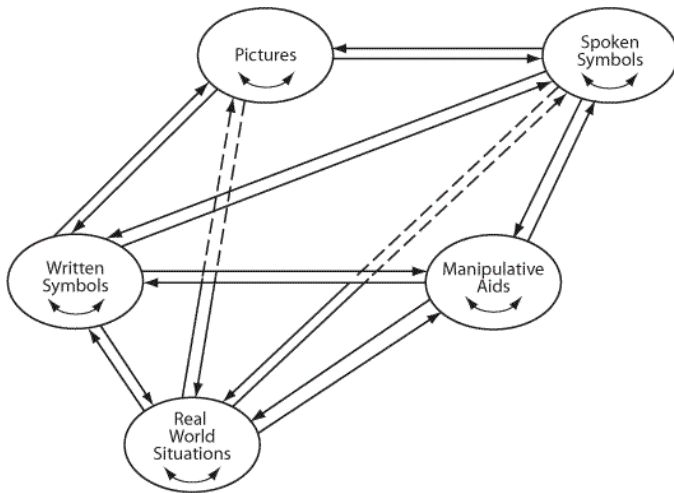
Ref: Behr et al 1980, 1981

Situations for fraction representation

Situations must be meaningful and hold children's interest.

Situations that work well

- Equal sharing situations (Streefland)
- Measure situations: flexible choice of unit (Lamon, Maher, Post)
- Measuring (in)completeness: work done, time elapsed, portion consumed, marks obtained, etc. (Moss 99, traditional?)

Why do these situations work?

May be triggering and developing pre-existing schemes (equi-partitioning, unitizing).

The fraction subconstructs

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Research
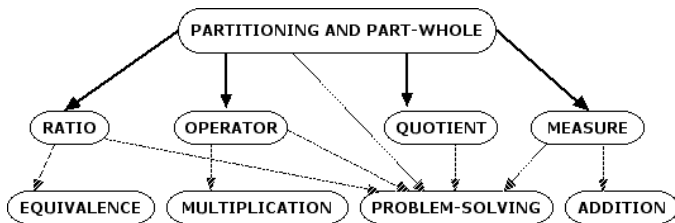Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
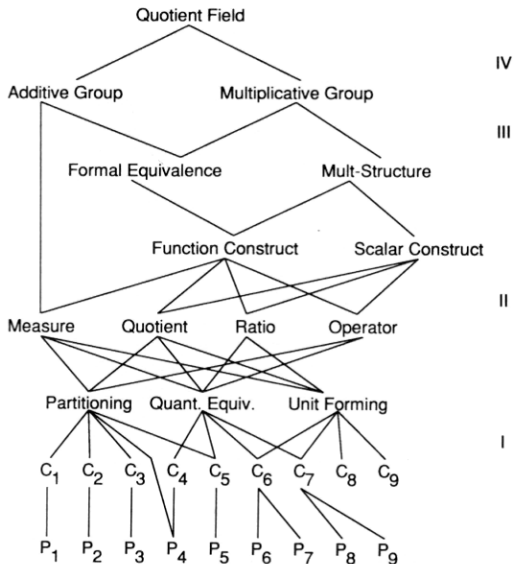multiplication
and division

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Reasoning

- Subconstruct analysis proposed by Thomas Kieren in the late 1970s.
- Fraction is not one concept, but is multi-faceted. Different subconstructs: part-whole, ratio, quotient, measure and operator.
- Analysis has been influential, but some have contested.
- Was part of the framework for the 'Rational Number Project' (1980s and 1990s). Behr et al. It "has stood the test of time."
- Useful framework to plan and design instruction.

RNP's model of subconstructs

Fractions
Research
Overview Mar
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K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

Ref: Behr et al 1980, 1981

Kieren's model of rational number thinking

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

Connections and Differences between the subconstructs – 1

- Part-whole is a special case of ratio where both quantities are from the same set.
- Addition and subtraction makes sense for measure intpn, not for ratio.
- But ordering/ comparison makes sense for ratio. (Concentration of mixtures, map scales, which rectangle is more oblong: 4:3 or 16:9?)
- Multiplication makes sense for ratios. It makes sense for measures only when it is appropriate from a physical viewpoint.

- Unit fraction concept makes sense for measure. Does it apply also to ratios?
- Quotient intpn: Partitive division is easy to grasp. But quotitive division takes us to the ratio interpretation in the sense of measure.
- Operator can be within measure space or between measure spaces. The semantics are different in the two cases.
- Ratio can be interpreted in different ways.
  - As pure ratio (e.g., length:breadth)
  - As measure of the first magnitude using the second as a unit (quotitive division).
  - As a multiplicative relation, i.e., in terms of a multiplier.

How does subconstruct theory help?

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

- Different subconstructs strengthen conceptual understanding of fractions and operations on fractions.
- These are important in the application of fractions to situations.
- Students need to become familiar with the multiple meanings of fractions. Subconstruct theory helps curriculum designers implement this.

Question: Does understanding of multiple subconstructs help in reasoning about ratio and proportion?

- Originates with Piaget.
- Developed by Leslie Steffe and his colleagues.
- Fraction schemes develop out of sophisticated counting schemes.
- Many schemes have been described, analysed and applied to understand children's actions.
- For example, Nabors 03 has used schemes to identify connections between reasoning about fractions and about proportion.

- The equipartitioning scheme for making fair shares
- The splitting scheme: creating equal parts or copies of an original. Focus on the one-to-many action (multiplicative), rather than on iterating a part or subunit (additive) – Confrey.
- But is splitting a composition of paritioning and iterating? (Steffe 03)
- Partitive unit fraction scheme: Partitioning to create an iterable fractional unit. Preserves the one-to-many relation between part and whole (e.g. one-tenth.)

Example: A proportion scheme

The units coordination scheme - Nabors, 03

3 pounds for 4 dollars; how many pounds for 28 dollars?

- Build-up strategy: 8 & 6, 12 & 9, ... (composite units coordinating scheme)
- Abbreviated build-up strategy: Anticipation: 28 is so many fours. 28 is 7 fours... 7 threes are 21 ... (iterable composite units coordinating scheme)
- Ratio or rate *construct*: 3/4 pounds to the dollar. (not a scheme?)
- Solution by equal ratios/rates: several schemes including fraction reducing scheme

Schemes create anticipations prior to action.

Discrete vs continuous wholes

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

Discrete whole

- Natural Units are not the 'whole' under consideration
- Counting
- Operator subconstruct to the fore
- Flexibility possible in basic unit and subunits

Continuous whole

- Natural Unit is the 'whole' under consideration
- Measuring operation
- Measure subconstruct to the fore
- Flexibility possible only in subunits

You could also have a mixed situation: multiple continuous wholes.

Discrete vs continuous wholes

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

- So working with continuous wholes emphasizes the measure subconstruct, while working with discrete wholes emphasizes the operator subconstruct.
- Studies dealing with discrete wholes: Saenz-Ludlow 03
- Studies dealing with continuous wholes: Nabors 03, Steenken and Maher, 03

- Small scale researcher taught teaching expts generally report good student gains in fraction ordering, representation, informal addn/subn.
- Theoretical issues: How best does one describe growing fraction understanding? strategies, schemes, subconstructs?
  - Elaboration of partitioning schems in different contexts (e.g. Empson et al.): continuous, discrete, discrete-continuous
  - Does the theory help develop better assmt tools?
  - Assmt tools that yield reliable information about student understanding: can they discriminate different strategies/ schemes / subconstructs?
  - Do these theoretical constructs help in strengthening teachers' knowledge and in better planning of teaching and design of learning material?

- Different kinds of studies are needed – larger in scale.
- Multiplication and division operations with fractions.
- Connection between fraction knowledge and understanding of ratio and proportion.

Fraction multiplication and division

There are very few studies.

Two studies by Mack (2000 and 2001): Studied the role of informal knowledge related to partitioning in meaningfully solving problems on multiplication of fractions.

Suggests that knowledge of partitioning is important. Complex knowledge about flexible unitizing may not be critical.

Fraction multiplication

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

A teaching study by Taber with fifth graders (1999 and 2002)

Classification of problem types:

Whole number/ fractional multiplier and whole number/fractional quantity

- Combine equal groups (ww and wf only)
- Multiplicative comparison (how many times?)
- Change (Alice story - stretching/shrinking)
- Partitioning (operating) (fw and ff only)

Begin with problems that are closer to whole number multiplication situations rather than partitioning problems.

Proportional Reasoning

- Was studied in the Piagetian tradition.
- Many studies upto the 80s on categorization of tasks, variables affecting performance and identifying student ability levels.
- Noelting 80: Development of proportional reasoning and ratio concept: Strictly Piagetian analysis based on an experimental orange juice task - 321 subjects from 6 to 16 years of age. Separated into Piagetian stages: IA to IIIB. Also discusses strategies (within and between sets) at each stage.

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

- Four main types of tasks : physical situations such as balance beam or length of shadows (Piaget), Missing value or rate problems, numerical comparison and qualitative comparison (RNP)
- Tourniaire and Pulos 85: Proportional reasoning: a review of literature. Gives a overview of correct and incorrect strategies and important variables affecting proportional reasoning. Says teaching studies are very few.
- Strategies to solve missing value problems: Unit rate, factor of change, algorithm and fraction strategies. (Cramer et al. 93)

- RNP stressed the importance of qualitative proportional reasoning and developed suitable tasks. (Heller et al. 1990)
- RNP teaching experiment on proportional reasoning.(Cramer et al. 1993)
- Nabors 03: Four Grade 7 students: 15 week teaching expt. Used a schemes analysis to study connection between reasoning on fraction problems and proportion problems.
- Moss 99: A 4th grade teaching expt starting with percent and developing rational number and proportional understanding.

Proportional Reasoning: RNP teaching experiment

Fractions
Research
Overview Mar
08

K.
Subramaniam

Overview of
issues

Initial Fraction
Knowledge

Fraction
multiplication
and division

Proportional
Reasoning

- Initially physical experiments with proportional and nonproportional situations. Students collected data, built tables, and determined the functional rule.
- Proportional situations were defined as those whose rule could be expressed in the form $y = mx$.
- Coordinate graphs were used to depict the data.
- Problem contexts evolved from familiar to less familiar ones
- Strategies taught: building tables to see number patterns, a unit-rate approach, a factor of change approach, and a fraction approach
- Unit-rate strategy was stressed initially.
- Standard algorithm (cross-multiply and divide) postponed until more meaningful, although less efficient, strategies were developed.

- Students learnt to solve problems using different strategies.
- Differences in preferred strategy. No one strategy preferred by all students.
- Even with the use of calculators students had trouble with noninteger relationships and often used additive strategies.
- Students using a unit-rate approach often had difficulty determining which unit rate to use as a factor. For example, 3 apples for 24 cents can be interpreted as 8 cents per apple or as 1/8 apple for 1cent.
- One strategy was to use a calculator to generate both unit rates, propose two answers, and then reason from the context of the problem as to which answer was most reasonable.